

SMA_UM: User Material Subroutine for Thermomechanical Constitutive Model of Shape Memory Alloys

Dimitris C. Lagoudas
Zhonghe Bo
Muhammad A. Qidwai
Pavlin B. Entchev

*Department of Aerospace Engineering
Texas A&M University
3141 TAMU
College Station, TX 77843-3141*

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1 Introduction

This manual provides a comprehensive review of user subroutine **SMA_UM** (**S**hape **M**emory **A**lloy — **U**nified **M**odel). **SMA_UM** is a FORTRAN coded numerical implementation of an SMA thermomechanical constitutive model. The current version of **SMA_UM** implements the unified constitutive model presented by Lagoudas et al. (1996), which unifies and generalizes to three dimensions the constitutive models presented by Tanaka (1986), Boyd and Lagoudas (1996) and Liang and Rogers (1992). The model is numerically implemented using return mapping algorithms. Detailed description of the implementation is provided in the research paper by Qidwai and Lagoudas (2000). The current implementation of the subroutine follows the specifications for user-material subroutines by ABAQUS and is intended primarily for use with ABAQUS. However, the subroutine can be integrated in any other standard finite element or computational program. The subroutine can be used in 3-D, 2-D plane strain and generalized plane strain, and 1-D problems.

A brief outline on the functions performed by the subroutine and the input and output parameters is given in Section 2. The capabilities of the subroutine are demonstrated in Section 3, where the results for various loading cases are presented.

2 Description of the Subroutine

2.1 Overview

In a Finite Element computational program the the analysis of materials with non-linear behavior usually involves the application of the Newton's method. During the application of Newton's method, the global stiffness matrix must be calculated by summing the entries of the local stiffness matrices for each integration point. To calculate these entries, the main computational program calls the material subroutine, where the local stiffness matrix is calculated according to the constitutive model for the material. Thus, the material subroutine is called for each integration point of the FE model at each iteration of the global Newton's method. When the subroutine is called, it is provided with the material state at the start of the increment such as stress, the solution dependent state variables, the temperature, the strain, etc., along with the temperature and strain increments. The output of the subroutine is the updated value of the stress tensor and the tangent stiffness tensor, as well as the values of all solution-dependent (internal) variables, such as the martensitic volume fraction and the transformation strain. The output provided by `SMA_UM` may not entirely be required for the successive operations of the calling program, but some of it may be demanded by the user for post-processing.

2.2 Unified Constitutive Model for SMAs

The unified constitutive model for SMAs, which unifies three thermodynamic constitutive models for SMAs is briefly presented here. The model has been presented in detail by Lagoudas et al. (1996).

The total Gibbs free energy is given by:

$$G(\boldsymbol{\sigma}, T, \xi, \boldsymbol{\varepsilon}^t) = -\frac{1}{2} \frac{1}{\rho} \boldsymbol{\sigma} : \mathbf{S} : \boldsymbol{\sigma} - \frac{1}{\rho} \boldsymbol{\sigma} : [\boldsymbol{\alpha}(T - T_0) + \boldsymbol{\varepsilon}^t] + c \left[(T - T_0) - T \ln \left(\frac{T}{T_0} \right) \right] - s_0 T + u_0 + f(\xi), \quad (1)$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}^t$, ξ , T and T_0 are the Cauchy stress tensor, transformation strain tensor, martensitic volume fraction, current temperature and reference temperature, respectively. The material constants \mathbf{S} , $\boldsymbol{\alpha}$, ρ , c , s_0 and u_0 are the effective compliance tensor, effective thermal expansion tensor, density, effective specific heat, effective entropy at reference state and effective specific internal energy at reference state. The above material constants are defined using the rule of mixtures as

$$\begin{aligned} \mathbf{S} &= \mathbf{S}^A + \xi(\mathbf{S}^M - \mathbf{S}^A), \quad \boldsymbol{\alpha} = \boldsymbol{\alpha}^A + \xi(\boldsymbol{\alpha}^M - \boldsymbol{\alpha}^A), \\ c &= c^A + \xi(c^M - c^A), \quad s_0 = s_0^A + \xi(s_0^M - s_0^A), \quad u_0 = u_0^A + \xi(u_0^M - u_0^A), \end{aligned} \quad (2)$$

where the quantities with the superscripts A and M indicate the values in the austenitic and the martensitic phases, respectively. The function $f(\xi)$ is

the transformation hardening function. By choosing the functional form of $f(\xi)$ different SMA constitutive models can be obtained, as described by Lagoudas et al. (1996).

The total strain $\boldsymbol{\varepsilon}$ is given by

$$\boldsymbol{\varepsilon} = \mathbf{S} : \boldsymbol{\sigma} + \boldsymbol{\alpha}(T - T_0) + \boldsymbol{\varepsilon}^t. \quad (3)$$

The relation between the transformation strain tensor $\boldsymbol{\varepsilon}^t$ and the martensitic volume fraction ξ is expressed by

$$\dot{\boldsymbol{\varepsilon}}^t = \mathbf{\Lambda} \dot{\xi}, \quad (4)$$

where $\mathbf{\Lambda}$ is the transformation tensor which determines the transformation strain direction. Two different forms of the transformation tensor $\mathbf{\Lambda}$ are implemented. The first is:

$$\mathbf{\Lambda} = \begin{cases} \frac{3}{2} H \frac{\boldsymbol{\sigma}'}{\bar{\sigma}}, & \dot{\xi} > 0 \\ H \frac{\boldsymbol{\varepsilon}^{t-r}}{\bar{\varepsilon}^{t-r}}, & \dot{\xi} < 0 \end{cases} \quad (5)$$

H is the maximum uniaxial transformation strain, $\boldsymbol{\varepsilon}^{t-r}$ is the transformation strain at the reversal of the transformation and

$$\bar{\sigma}^m = \sqrt{\frac{3}{2}} \|\boldsymbol{\sigma}^{m'}\|, \quad \boldsymbol{\sigma}^{m'} = \boldsymbol{\sigma}^m - \frac{1}{3} \text{tr}(\boldsymbol{\sigma}^m) \mathbf{I}, \quad \bar{\varepsilon}^{t-r} = \sqrt{\frac{2}{3}} \|\boldsymbol{\varepsilon}^{t-r}\|. \quad (6)$$

The transformation tensor given by equation (5) is suitable for proportional loading cases (e.g., uniaxial loading). For more complicated loading cases, the following form of the transformation tensor $\mathbf{\Lambda}$, independent of the transformation direction, is also implemented:

$$\mathbf{\Lambda} = \frac{3}{2} H \frac{\boldsymbol{\sigma}'}{\bar{\sigma}}. \quad (7)$$

It is recommended that the form of the transformation tensor given by equation (5) is used for computations. The form of $\mathbf{\Lambda}$ given by equation (7) should be used if convergence problems are experienced.

The thermodynamic force conjugate to ξ is given by

$$\begin{aligned} \pi = & \boldsymbol{\sigma} : \mathbf{\Lambda} + \frac{1}{2} \boldsymbol{\sigma} : \Delta \mathbf{S} : \boldsymbol{\sigma} + \Delta \boldsymbol{\alpha} : \boldsymbol{\sigma} (T - T_0) - \rho \Delta c \left[(T - T_0) - T \ln \left(\frac{T}{T_0} \right) \right] \\ & + \rho \Delta s_0 T - \frac{\partial f}{\partial \xi} - \rho \Delta u_0 \end{aligned} \quad (8)$$

where $f(\xi)$ is the hardening function, the terms with the prefix Δ indicate the difference of a quantity between the martensitic and austenitic phases and are given by

$$\begin{aligned} \Delta \mathbf{S} &= \mathbf{S}^M - \mathbf{S}^A, \quad \Delta \boldsymbol{\alpha} = \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^A, \quad \Delta c = c^M - c^A, \\ \Delta s_0 &= s_0^M - s_0^A, \quad \Delta u_0 = u_0^M - u_0^A. \end{aligned} \quad (9)$$

The transformation function Φ is defined in terms of the thermodynamic force π as

$$\Phi = \begin{cases} \pi - Y^*, & \dot{\xi} > 0 \\ -\pi - Y^*, & \dot{\xi} < 0 \end{cases} \quad (10)$$

where Y^* is the measure of internal dissipation due to the phase transformation. Constraints on the evolution of the martensitic volume fraction are expressed in terms of the Kuhn-Tucker conditions as

$$\begin{aligned} \dot{\xi} &\geq 0, & \Phi(\boldsymbol{\sigma}, T, \xi) &\leq 0, & \Phi\dot{\xi} &= 0, \\ \dot{\xi} &\leq 0, & \Phi(\boldsymbol{\sigma}, T, \xi) &\leq 0, & \Phi\dot{\xi} &= 0. \end{aligned} \quad (11)$$

As mentioned earlier, different forms of the transformation hardening function $f(\xi)$ can be selected to recover different constitutive models. For the exponential model reported by Tanaka (1986) the function $f(\xi)$ can be selected as

$$f(\xi) = \begin{cases} \frac{\Delta s_0}{a_e^M} [(1 - \xi) \ln(1 - \xi) + \xi] + (\mu_1^e + \mu_2^e)\xi, & \dot{\xi} > 0, \\ -\frac{\Delta s_0}{a_e^A} \xi [\ln(\xi) - 1] + (\mu_1^e - \mu_2^e)\xi, & \dot{\xi} < 0, \end{cases} \quad (12)$$

while for the cosine model presented by Liang and Rogers (1992) the function takes the form

$$f(\xi) = \begin{cases} \int_0^\xi -\frac{\Delta s_0}{a_c^M} [\pi - \cos^{-1}(2\tilde{\xi} - 1)] d\tilde{\xi} + (\mu_1^c + \mu_2^c)\xi, & \dot{\xi} > 0, \\ \int_0^\xi -\frac{\Delta s_0}{a_c^A} [\pi - \cos^{-1}(2\tilde{\xi} - 1)] d\tilde{\xi} + (\mu_1^c - \mu_2^c)\xi, & \dot{\xi} < 0. \end{cases} \quad (13)$$

To obtain the constitutive model presented by Boyd and Lagoudas (1996) the function $f(\xi)$ is selected as

$$f(\xi) = \begin{cases} \frac{1}{2}\rho b^M \xi^2 + (\mu_1^p + \mu_2^p)\xi, & \dot{\xi} > 0, \\ \frac{1}{2}\rho b^A \xi^2 + (\mu_1^p - \mu_2^p)\xi, & \dot{\xi} < 0. \end{cases} \quad (14)$$

The quantities a_e^M , a_e^A , μ_1^e , a_c^M , a_c^A , μ_1^c , b^M , b^A and μ_1^p are material constants, while μ_2^e , μ_2^c and μ_2^p are introduced to enforce continuity for the function $f(\xi)$ during forward and reverse transformation.

2.3 Material Parameters

The following material parameters are required by the **SMA_UM** subroutine: the Young's moduli of both austenite and martensite E^A and E^M , thermal expansion coefficients α^A and α^M , martensite start and finish and austenite start and finish temperatures at zero stress M^{0s} , M^{0f} , A^{0s} and A^{0f} , maximum transformation strain H and austenite and martensite stress influence coefficients $\rho\Delta s^A$ and $\rho\Delta s^M$. In addition, depending on the model used, the hardening constants must also be specified. The procedure for the determination of the material constants is briefly described below.

The elastic Young's moduli of the austenite and martensite can be obtained from a uniaxial pseudoelastic test. The elastic stiffness of the austenite can be determined by measuring the slope of the stress-strain curve at the beginning of the loading while the elastic stiffness of the martensite can be determined by measuring the slope at the beginning of the unloading (see Figure 1). Thermal expansion coefficients can be obtained by performing standard test at lower temperature for the martensite and at higher temperature for the austenite. The transformation temperatures at zero stress can be determined from a Differential Scanning Calorimeter (DSC) test. The maximum transformation strain can be obtained from a pseudoelastic test, as shown in Figure 1. The stress influence coefficients $\rho\Delta s^A$ and $\rho\Delta s^M$ can be determined from the stress-temperature phase diagram and a pseudoelastic test. These parameters can be calculated as

$$\rho\Delta s^A = -\frac{\sigma^A}{T_{\text{test}} - A^{0s}}H, \quad (15)$$

$$\rho\Delta s^M = -\frac{\sigma^M}{T_{\text{test}} - M^{0s}}H, \quad (16)$$

where T_{test} is the temperature at which the pseudoelastic test was performed and σ^A and σ^M are defined in Figure 1.

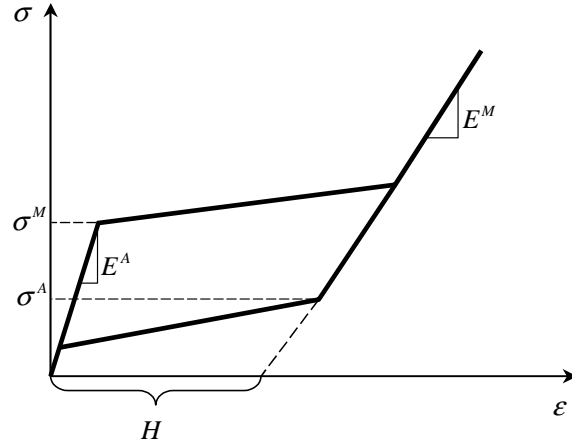


Figure 1: Schematic of an SMA uniaxial pseudoelastic test.

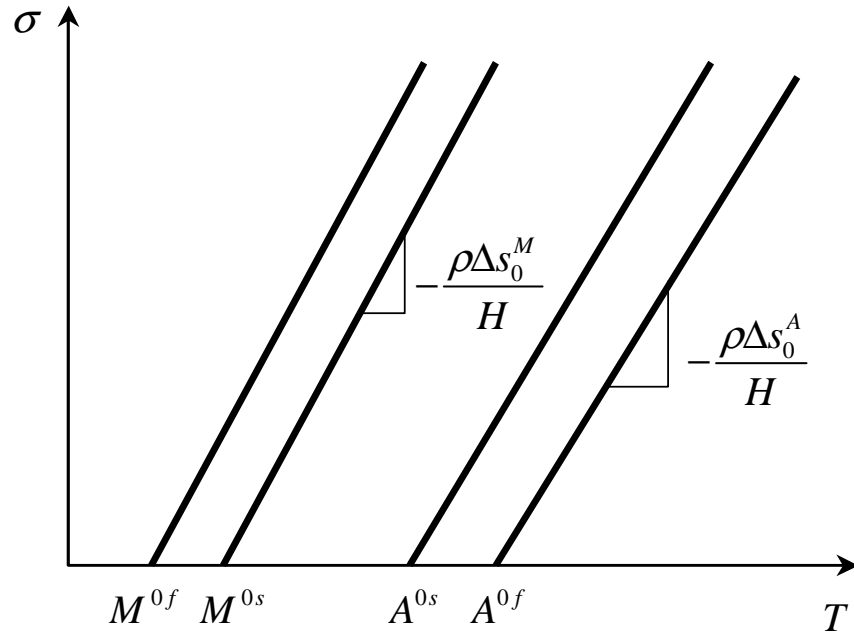


Figure 2: Schematic of an SMA phase diagram.

2.4 Definitions of Input and Output Variables for the Subroutine

The subroutine is declared according to the rules for User-Material Subroutines provided by ABAQUS. The following three sets of lines are written according to the ABAQUS guidelines:

```

SUBROUTINE UMAT(STRESS,STATEV,DDSDDE,SSE,SPD,SCD,
1RPL,DDSDDT,DRPLDE,DRPLDT,
2STRAN,DSTRAN,TIME,DTIME,TEMP,DTEMP,PRED,DPRED,CMNAME,
3NDI,NSHR,NTENS,NSTATV,PROPS,NPROPS,COORDS,DROT,PNEWDT,
4CELENT,DFGRD0,DFGRD1,NOEL,NPT,LAYER,KPST,KSTEP,KINC)
C
INCLUDE 'ABA_PARAM.INC'
C
CHARACTER*8 CMNAME
DIMENSION STRESS(NTENS),STATEV(NSTATV),
1DDSDDE(NTENS,NTENS),DDSDDT(NTENS),DRPLDE(NTENS),
2STRAN(NTENS),DSTRAN(NTENS),TIME(2),PRED(1),DPRED(1),
3PROPS(NPROPS),COORDS(3),DROT(3,3),DFGRD0(3,3),DFGRD1(3,3)

```

Not all of the parameters are utilized in the current version of SMA_UM. The relevant input and output parameters are discussed below.

The following input parameters are passed from the calling program to the subroutine:

STRAN(NTENS)	An array containing the components of the total strain tensor at the beginning of the increment
DSTRAN(NTENS)	An array containing the increment of the strain tensor
TIME(1)	Value of the step time at the beginning of the current increment for the global Newton's method
TIME(2)	Value of total time at the beginning of the current increment
TEMP	Temperature at the start of the increment
DTEMP	Increment of temperature
NDI	Number of direct stress components at the current integration point
NSHR	Number of engineering shear stress components at the current integration point
NTENS	Size of the stress or strain component array (NDI + NSHR)
NSTATV	Number of solution dependent state variables
NPROPS	Number of material constants
PROPS(NPROPS)	Array of material constants used by the subroutine

The following output variables are returned by the subroutine to the calling program:

DDSDDE(NTENS, NTENS)	An array containing the components of the tangent stiffness matrix $\partial\sigma/\partial\epsilon$
STRESS(NTENS)	An array containing the components of the stress tensor at the beginning of the increment, and the updated components at the end of the increment (input/output)
STATEV(NSTATV)	An array containing the solution dependent state variables. These values are provided to the subroutine at the beginning of the increment as input and are updated and returned at the end of the increment as output

3 Example on the Usage of the Subroutine in ABAQUS environment

To demonstrate the capabilities of the subroutine, an example problem is presented in this section. The analyses are performed using ABAQUS, where the material response of the SMA is described using SMA_UM.

3.1 Setup of the Material Parameters

The material parameters for NiTi SMA used in the analyses are taken from the work of Qidwai and Lagoudas (2000) and are shown in Table 1. The portion of the input file with the definitions of the material parameters is also shown below.

Table 1: Material parameters for NiTi SMA.

Material Parameter	Value
Elastic stiffness of the austenite E^A	70 GPa
Elastic stiffness of the martensite E^M	30 GPa
Poisson's ratio (equal for both phases) ν	0.33
Coefficient of thermal expansion for the austenite α^A	$22.0 \times 10^{-6} \text{ K}^{-1}$
Coefficient of thermal expansion for the martensite α^M	$22.0 \times 10^{-6} \text{ K}^{-1}$
Martensitic start temperature M^{0s}	291 K
Martensitic finish temperature M^{0f}	271 K
Austenitic start temperature A^{0s}	295 K
Austenitic finish temperature A^{0f}	315 K
Maximum transformation strain H	0.05
Stress influence coefficient for austenite $\rho\Delta s^A$	-0.35 MPa K ⁻¹
Stress influence coefficient for martensite $\rho\Delta s^M$	-0.35 MPa K ⁻¹

Definition of the material parameters in the input file:

```
*MATERIAL, NAME=SMA
```



```

*DENSITY
6450.0
*SPECIFIC HEAT
329.0
*CONDUCTIVITY
22.0
*DEPVAR
100
*USER MATERIAL, CONSTANTS=24
1.0, 2.0, 1.0E-8, 0.0, 16.0, 70.0E9, 30.0E9, 0.33,
22.0E-6, 22.0E-6, 291.0, 271.0, 295.0, 315.0, 0.05, -0.35E6,
-0.35E6, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1.0

```

The material definition is symbolically given below, and each of the parameters is explained. First, the number of solution-dependent state variables is given after the keyword `*DEPVAR`. Next, the input parameters for the subroutine are given under the keyword `*USER MATERIAL`, where the number of the material constant is given after the keyword `*CONSTANTS`. Due to the ABAQUS requirements, the parameters are given in groups of 8 entries per line. The following parameters are defined:

```

IPHASE,MODEL,TOL,xi0,NELMTP,EA,EM,nu,
alphaA,alphaM,Mos,Mof,Aos,Aof,H,rDs0A,
rDs0M,epstr11,epstr22,epstr33,2epstr23,2epstr13,2epstr12,FRULE

```

The definitions for the parameters in the material definition section are given below. Note that the material parameters are expressed in SI units. Any consistent units system may be used for the mechanical parameters. However, the in parameters involving temperature, the temperature must be expressed in Kelvin.

IPHASE	The phase of the material: 1 – austenite, 2 – martensite
MODEL	The constitutive model: 1 — Tanaka’s exponential model, 2 — Boyd and Lagoudas’ polynomial model, 3 — Liang and Rogers’ cosine model
TOL	Convergence criterion tolerance: the iteration process terminates if the absolute value of the increment of the martensitic volume fraction is less than TOL
xi0	Initial value of the martensitic volume fraction
NELMTP	Number of integration points in all SMA finite elements
EA	Young’s modulus for austenite, Pa
EM	Young’s modulus for martensite, Pa
nu	Poisson’s ratio
alphaA	Thermal expansion coefficient of austenite, K^{-1}
alphaM	Thermal expansion coefficient of martensite, K^{-1}
Mos	Martensitic start temperature, K
Mof	Martensitic finish temperature, K
Aos	Austenitic start temperature, K
Aof	Austenitic finish temperature, K
H	Maximum transformation strain
rSo/H	Martensitic stress influence coefficient, $Pa\ K^{-1}$
rSo/H	Austenitic stress influence coefficient, $Pa\ K^{-1}$
epstr11	Initial value of the 11-component of the transformation strain tensor
epstr22	Initial value of the 22-component of the transformation strain tensor
epstr33	Initial value of the 33-component of the transformation strain tensor
2epstr23	Twice the initial value of the 23-component of the transformation strain tensor
2epstr13	Twice the initial value of the 13-component of the transformation strain tensor
2epstr12	Twice the initial value of the 12-component of the transformation strain tensor
FRULE	Flag for the form of the transformation tensor $\mathbf{\Lambda}$: 1 — given by equation (5), 2 — given by equation (7)

The solution dependent state variables for each integration point are stored in a designated array and are available for postprocessing. The following state variables may be requested:

SDV1	Direction of transformation indicator: 1 for forward and 2 for reverse
SDV2	Martensitic volume fraction
SDV3	Flag for transformation: 0 for no transformation, 1 for transformation
SDV5	Component of the transformation strain in the 11 direction
SDV6	Component of the transformation strain in the 22 direction
SDV7	Component of the transformation strain in the 33 direction
SDV8	Twice the value of the transformation strain in the 23 direction
SDV9	Twice the value of the transformation strain in the 13 direction
SDV10	Twice the value of the transformation strain in the 12 direction
SDV14	Value of the effective transformation strain $\bar{\epsilon}^t = \sqrt{\frac{2}{3}\epsilon^t : \epsilon^t}$
SDV15	Current temperature
SDV20	Component of the transformation direction tensor \mathbf{A} in the 11 direction
SDV21	Component of the transformation direction tensor \mathbf{A} in the 22 direction
SDV22	Component of the transformation direction tensor \mathbf{A} in the 33 direction
SDV23	Twice the value of the component of the transformation direction tensor \mathbf{A} in the 23 direction
SDV24	Twice the value of the component of the transformation direction tensor \mathbf{A} in the 13 direction
SDV25	Twice the value of the component of the transformation direction tensor \mathbf{A} in the 12 direction
SDV28	Value of the martensitic volume fraction at the point of reversal of the transformation

3.2 Uniaxial Loading Cases

A three-dimensional unit cube ($1\text{ m} \times 1\text{ m} \times 1\text{ m}$) of SMA is subjected to uniaxial loading. The cube is divided into two finite elements, as shown in Figure 3. The elements used in this example are eight-node brick elements C3D8. Initially the material is nickel-titanium shape memory alloy in the austenitic state. Different loading conditions are applied, as described below:

- Test 1 The first loading case demonstrates the capabilities of the subroutine to describe the pseudoelastic response of the material. The material is at a temperature higher than the austenitic finish temperature A^{0f} and the loading is applied uniaxially until full transformation is achieved. The material is then unloaded to zero stress, thus recovering all of the transformation strain.
- Test 2 The second loading case is a demonstration of the Shape Memory Effect (SME). Initially the material is at a temperature between the austenitic start A^{0s} and the martensitic start M^{0s} . Uniaxial stress loading is ap-

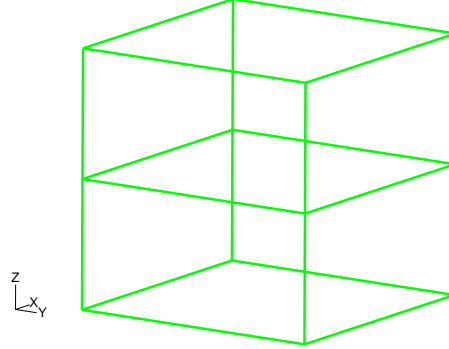


Figure 3: Finite element mesh for the test problem.

plied, which induces martensitic phase transformation. Upon unloading, the material remains in the martensitic phase and at zero stress there is residual strain. Next, the material is heated to temperature higher than A^{0f} . During heating, the material undergoes reverse phase transformation and the strain is completely recovered.

Test 3 The third loading case demonstrated the capabilities of the subroutine to describe temperature-induced phase transformation with applied stress. Initially the material is at temperature above A^{0f} . Uniaxial stress loading is applied such that the value of the applied stress is not sufficient to induce martensitic phase transformation. Next, holding the value of the applied stress constant, the material is cooled to a temperature below martensitic finish M^{0f} . During cooling the material undergoes phase transformation and large strains are observed. Next, the material is heated back to the initial temperature, which results in reverse phase transformation and recovery of the transformation strain.

The stress-strain response of the SMA material for the first test case is shown in Figure 4. The results were obtained using all three constitutive models: the exponential model (Tanaka, 1986), the cosine model (Liang and Rogers, 1992) and the model with a polynomial hardening function (Boyd and Lagoudas, 1996). The numerical simulations were performed at a temperature of 325 K.

The results for the second loading case, demonstrating the SME are shown in Figure 5. As in the previous loading case, the results obtained with all three of the models are shown.

The temperature-strain response of NiTi SMA is shown in Figure 6. The applied tensile load for this case was equal to 100 MPa.

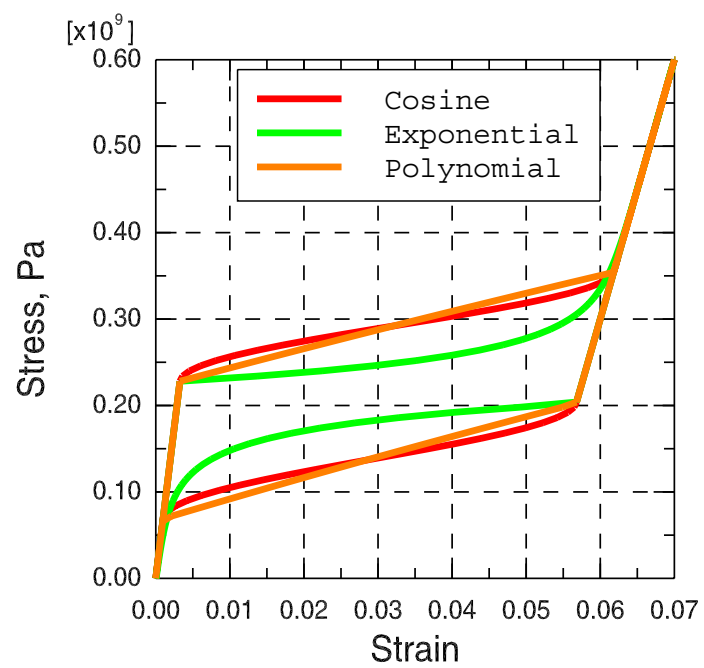


Figure 4: Pseudoelastic response of NiTi SMA.

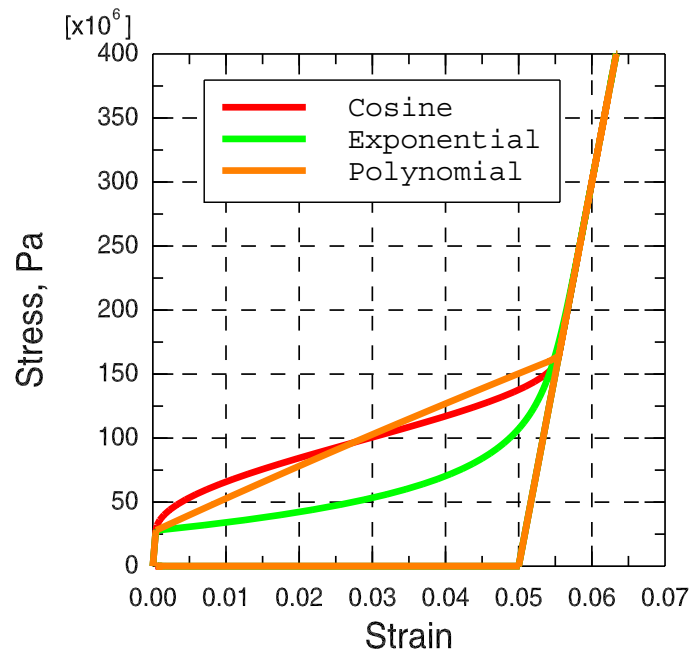


Figure 5: Shape Memory Effect response of NiTi SMA.

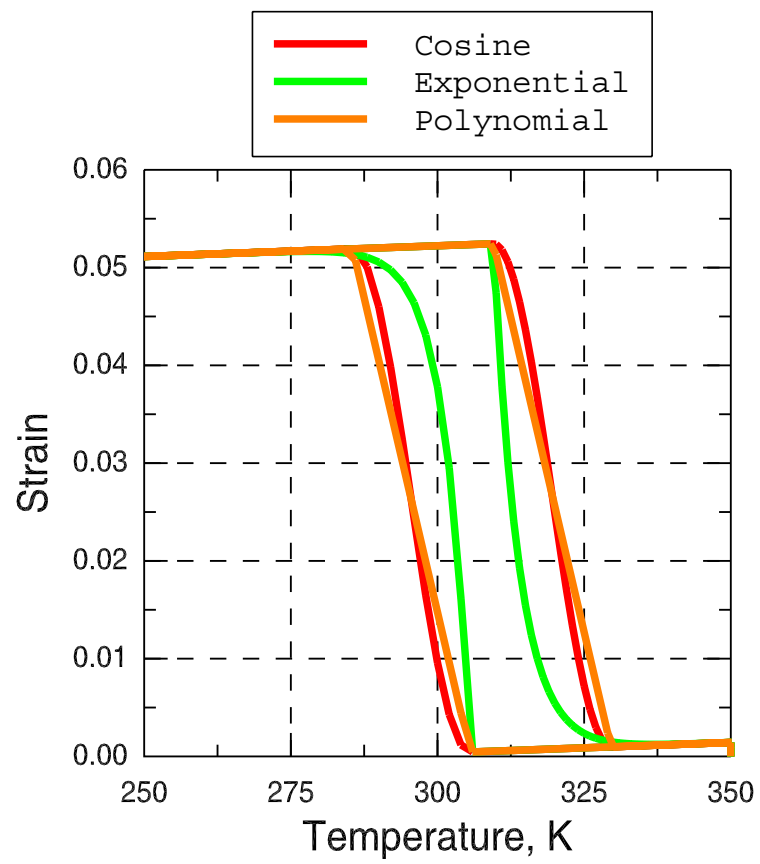


Figure 6: Thermally-induced phase transformation in NiTi SMA.

3.3 Loading of a Torque Tube

The capabilities of SMA_UM to handle loading cases beyond uniaxial loading are demonstrated by simulating an SMA torque tube. The following dimensions of the tube are used: outer diameter $d_o = 6.34$ mm and inner diameter $d_{in} = 5.0$ mm. The reason for selecting these dimensions is to model a tube which geometrically resembles tubes available commercially.¹

Based on the small thickness of the tube wall only one quadratic element in radial direction is used. In addition, since the stress is constant in the axial direction, one element in the axial direction is sufficient to obtain accurate results. To obtain appropriate aspect ratio, the length in the axial direction has been chosen to be 0.67 mm, equal to the wall thickness. An axisymmetric finite element with a rotational degree of freedom (element CGAX8 from the ABAQUS element library, see HKS, 1997) was used. The schematic of the mesh and the boundary conditions is shown in Figure 7. The maximum value of the applied rotation (see Figure 7) is taken to be $\theta^{\max} = 0.015$ rad ($\approx 0.86^\circ$), while the maximum value of the displacement in the axial direction is taken to be $u_z^{\max} = 0.0268$ mm, corresponding to axial tensile strain of $\varepsilon_{zz} = 0.04$. Both tension and rotation boundary conditions are applied on the top surface of the specimen, while the bottom surface is held fixed in the z - and θ -directions. Traction-free boundary condition in the radial direction is applied. The numerical simulations are performed at a temperature of 350 K. The model with a polynomial hardening function (Boyd and Lagoudas, 1996) was used to perform the analysis.

The results for sequential torsion-tension loading are presented. During the first loading step when the tube is subjected to torsion. As the stress increases the critical stress for the onset of the phase transformation is first reached at the outer surface of the tube. The phase transformation front then propagates towards the inner surface of the tube. During the second loading step, the tube is loaded in tension.

The history of the *average* axial and shear stress components during both loading steps is shown in Figure 8. During the initial torsional loading the shear stress linearly increases until a critical value is reached. At this point (at the value of the loading parameter of ≈ 0.12) the phase transformation initiates. During the first loading step the axial stress component remains zero. During the second loading step the phase transformation continues until it is fully completed (at the value of the loading parameter of ≈ 0.9). The axial stress increases while the shear stress is partially relaxed. After the completion of the phase transformation the material is fully in the martensitic phase and behaves linearly.

To explain the partial relaxation of the shear stress during the tensile axial loading, consider the shear and axial components of the transformation strain. During the second loading step, when a tensile loading is applied, the shear component of the transformation strain continues to increase. This is caused

¹The diameters used here have also been used by Qidwai and Lagoudas (2000) and are based on the specifications of torque tubes manufactured by Memry Corp.

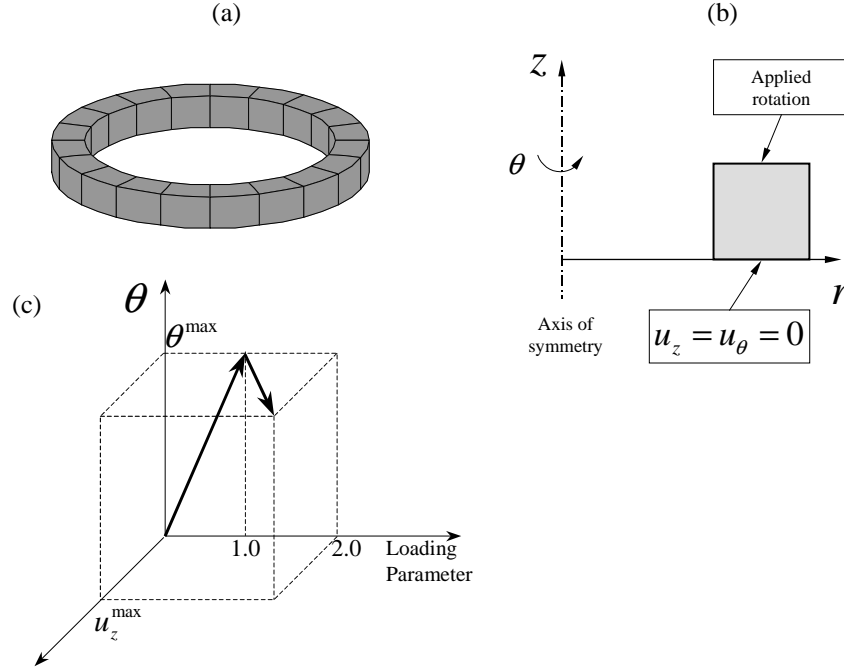


Figure 7: NiTi SMA torque tube: (a) geometry and finite element mesh; (b) boundary conditions; (c) loading history.

by the choice of the functional form of the transformation direction tensor $\mathbf{\Lambda}$, given by equation (5). Since the shear component of the stress is non-zero, the shear component of the transformation strain keeps developing during the second loading step, albeit with a smaller rate. On the other hand, since the applied rotation and, therefore, the total strain, is kept constant during the tensile loading, the shear component of the stress decreases according to the constitutive equation (3).

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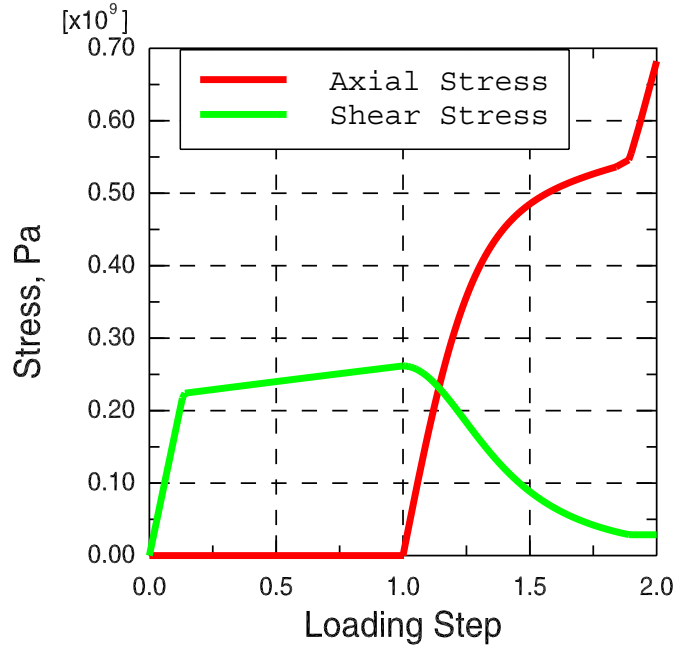


Figure 8: History of the axial and shear stress components during sequential torsion-tension loading.

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Qidwai, M. A., Lagoudas, D. C., 2000. On thermomechanics and transformation surfaces of polycrystalline NiTi shape memory alloy material. *Int. J. Plasticity* 16, 1309–1343.

Tanaka, K., 1986. A thermomechanical sketch of shape memory effect: One-dimensional tensile behavior. *Res Mechanica* 18, 251–263.

A Input File Listings

Listing 1: Input file for uniaxial pseudoelastic loading of NiTi SMA

```
*HEADING
Uniaxial Isothermal Mechanical Loading of SMA
*NODE, NSET=B1
1, 1.0, 0.0, 0.0
2, 1.0, 1.0, 0.0
3, 0.0, 1.0, 0.0
4, 0.0, 0.0, 0.0
*NODE, NSET=B2
9, 1.0, 0.0, 1.0
10, 1.0, 1.0, 1.0
11, 0.0, 1.0, 1.0
12, 0.0, 0.0, 1.0
*NODE, NSET=A3
5, 1.0, 0.0, 0.50
6, 1.0, 1.0, 0.50
7, 0.0, 1.0, 0.50
8, 0.0, 0.0, 0.50
*ELEMENT, TYPE=C3D8, ELSET=EALL
1, 1,2,3,4,5,6,7,8
2, 5,6,7,8,9,10,11,12
*SOLID SECTION, ELSET=EALL, MATERIAL=SMA
*MATERIAL, NAME=SMA
*DENSITY
6450.0
*SPECIFIC HEAT
329.0
*CONDUCTIVITY
22.0
*DEPVAR
100
*USER MATERIAL, CONSTANTS=24
1.0, 3.0, 1.0E-8, 0.0, 16, 70.0E9, 30.0E9, 0.33,
22.0E-6, 22.0E-6, 291.0, 271.0, 295.0, 315.0, 0.05, -0.35E6,
-0.35E6, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1.0
*BOUNDARY
B1,3
1,1,2
2,1
*NSET, NSET=NOUT
9
*ELSET, ELSET=ELOUT
1
*INITIAL CONDITIONS, TYPE=TEMPERATURE
B1, 325.0
B2, 325.0
A3, 325.0
```

```

*STEP, INC=1000
*STATIC
1.0,100.0,,1.0
*CLOAD
9,3,150.E6
10,3,150.E6
11,3,150.E6
12,3,150.E6
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=1, POSITION=CENTROIDAL, SUMMARY=NO
S11,S22,S33,S12,MISES,SDV2,SDV14,SDV15
*NODE PRINT, FREQUENCY=1, SUMMARY=NO, NSET=NOUT
CF,U
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL
TEMP, S, E
*ENDSTEP
*STEP, INC=1000
*STATIC
1.0,100.0,,1.0
*CLOAD
9,3,0.0
10,3,0.0
11,3,0.0
12,3,0.0
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=1, POSITION=CENTROIDAL, SUMMARY=NO
S11,S22,S33,S12,MISES,SDV2,SDV14,SDV15
*NODE PRINT, FREQUENCY=1, SUMMARY=NO, NSET=NOUT
CF,U
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL
TEMP, S, E
*ENDSTEP

```

Listing 2: Input file for uniaxial loading of NiTi SMA demonstrating the shape memory effect

```

*HEADING
Uniaxial Shape Memory Effect.
*NODE, NSET=B1
1, 1.0, 0.0, 0.0
2, 1.0, 1.0, 0.0
3, 0.0, 1.0, 0.0
4, 0.0, 0.0, 0.0
*NODE, NSET=B2
9, 1.0, 0.0, 1.0
10, 1.0, 1.0, 1.0
11, 0.0, 1.0, 1.0
12, 0.0, 0.0, 1.0

```

```

*NODE, NSET=A3
5, 1.0, 0.0, 0.50
6, 1.0, 1.0, 0.50
7, 0.0, 1.0, 0.50
8, 0.0, 0.0, 0.50
*ELEMENT, TYPE=C3D8, ELSET=EALL
1, 1,2,3,4,5,6,7,8
2, 5,6,7,8,9,10,11,12
*SOLID SECTION, ELSET=EALL, MATERIAL=SMA
*MATERIAL, NAME=SMA
*DENSITY
6450.0
*SPECIFIC HEAT
329.0
*CONDUCTIVITY
22.0
*DEPVAR
100
*USER MATERIAL, CONSTANTS=24
1.0, 3.0, 1.0E-8, 0.0, 16, 70.0E9, 30.0E9, 0.33,
22.0E-6, 22.0E-6, 291.0, 271.0, 295.0, 315.0, 0.05, -0.35E6,
-0.35E6, 0.0, 0.0, 0.0, 0.0, 0.0, 1.0
*BOUNDARY
B1,3
1,1,2
2,1
*NSET, NSET=NOUT
9
*ELSET, ELSET=ELOUT
1
*INITIAL CONDITIONS, TYPE=TEMPERATURE
B1, 295.0
B2, 295.0
A3, 295.0
*STEP, INC=1000
*STATIC
1.0,100.0,,1.0
*CLOAD
9,3,100.e6
10,3,100.e6
11,3,100.e6
12,3,100.e6
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=1, POSITION=CENTROIDAL, SUMMARY=NO
S11,S22,S33,S12,MISES,SDV2,SDV14,SDV15
*NODE PRINT, FREQUENCY=1, SUMMARY=NO, NSET=NOUT
CF,U
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL
TEMP, S, E

```

```

*ENDSTEP
*STEP, INC=1000
*STATIC
1.0,100.0,,1.0
*CLOAD
9,3,0.0
10,3,0.0
11,3,0.0
12,3,0.0
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=1, POSITION=CENTROIDAL, SUMMARY=NO
S11,S22,S33,S12,MISES,SDV2,SDV14,SDV15
*NODE PRINT, FREQUENCY=1, SUMMARY=NO, NSET=NOUT
CF,U
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL
TEMP, S, E
*ENDSTEP
*STEP, INC=1500
*STATIC
1.0,100.0,,1.0
*CONTROLS, PARAMETERS=LINE SEARCH
4
*TEMPERATURE
B1, 350
B2, 350
A3, 350
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=1, POSITION=CENTROIDAL, SUMMARY=NO
S11,S22,S33,S12,MISES,SDV2,SDV14,SDV15
*NODE PRINT, FREQUENCY=1, SUMMARY=NO, NSET=NOUT
CF,U
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL
TEMP, S, E
*ENDSTEP

```

Listing 3: Input file for uniaxial thermally-induced transformation of NiTi SMA

```

*HEADING
Uniaxial Loading of SMA with Thermally-Induced Phase Transf.
*NODE, NSET=B1
1, 1.0, 0.0, 0.0
2, 1.0, 1.0, 0.0
3, 0.0, 1.0, 0.0
4, 0.0, 0.0, 0.0
*NODE, NSET=B2
9, 1.0, 0.0, 1.0
10, 1.0, 1.0, 1.0
11, 0.0, 1.0, 1.0

```

```

12, 0.0, 0.0, 1.0
*NODE, NSET=A3
5, 1.0, 0.0, 0.50
6, 1.0, 1.0, 0.50
7, 0.0, 1.0, 0.50
8, 0.0, 0.0, 0.50
*ELEMENT, TYPE=C3D8, ELSET=EALL
1, 1,2,3,4,5,6,7,8
2, 5,6,7,8,9,10,11,12
*SOLID SECTION, ELSET=EALL, MATERIAL=SMA
*MATERIAL, NAME=SMA
*DENSITY
6450.0
*SPECIFIC HEAT
329.0
*CONDUCTIVITY
22.0
*DEPVAR
100
*USER MATERIAL, CONSTANTS=24
1.0, 3.0, 1.0E-8, 0.0, 16, 70.0E9, 30.0E9, 0.33,
22.0E-6, 22.0E-6, 291.0, 271.0, 295.0, 315.0, 0.05, -0.35E6,
-0.35E6, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1.0
*BOUNDARY
B1,3
1,1,2
2,1
*NSET, NSET=NOUT
9
*ELSET, ELSET=ELOUT
1
*INITIAL CONDITIONS, TYPE=TEMPERATURE
B1, 350.0
B2, 350.0
A3, 350.0
*STEP, INC=1000
*STATIC
1.0,100.0,,1.0
*CLOAD
9,3,10.e6
10,3,10.e6
11,3,10.e6
12,3,10.e6
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=1, POSITION=CENTROIDAL, SUMMARY=NO
S11,S22,S33,S12,MISES,SDV2,SDV14,SDV15
*NODE PRINT, FREQUENCY=1, SUMMARY=NO, NSET=NOUT
CF,U
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL

```

```

TEMP, S, E
*ENDSTEP
*STEP, INC=1100
*STATIC
2.0,100.0,,2.0
*CONTROLS, PARAMETERS=LINE SEARCH
4
*TEMPERATURE
B1, 250
B2, 250
A3, 250
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=1, POSITION=CENTROIDAL, SUMMARY=NO
S11, S22, S33, S12, MISES, SDV2, SDV14, SDV15
*NODE PRINT, FREQUENCY=1, SUMMARY=NO, NSET=NOUT
CF, U
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL
TEMP, S, E
*ENDSTEP
*STEP, INC=1500
*STATIC
1.0,100.0,,1.0
*CONTROLS, PARAMETERS=LINE SEARCH
4
*TEMPERATURE
B1, 350
B2, 350
A3, 350
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=1, POSITION=CENTROIDAL, SUMMARY=NO
S11, S22, S33, S12, MISES, SDV2, SDV14, SDV15
*NODE PRINT, FREQUENCY=1, SUMMARY=NO, NSET=NOUT
CF, U
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL
TEMP, S, E
*ENDSTEP
*STEP, INC=1000
*STATIC
1.0,100.0,,1.0
*CLOAD
9,3,0.0
10,3,0.0
11,3,0.0
12,3,0.0
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=1, POSITION=CENTROIDAL, SUMMARY=NO
S11, S22, S33, S12, MISES, SDV2, SDV14, SDV15
*NODE PRINT, FREQUENCY=1, SUMMARY=NO, NSET=NOUT

```



```

CF,U
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL
TEMP, S, E
*ENDSTEP

```

Listing 4: Input file for multiaxial loading of an SMA torque tube

```

*HEADING
Multiaxial Loading of an SMA Torque Tube
*NODE, NSET=NALL
1, 2.5E-3, 0.0
2, 3.17E-3, 0.0
3, 3.17E-3, 0.67E-3
4, 2.5E-3, 0.67E-3
5, 2.835E-3, 0.0
6, 3.17E-3, 0.335E-3
7, 2.835E-3, 0.67E-3
8, 2.5E-3, 0.335E-3
*ELEMENT, TYPE=CGAX8, ELSET=ELALL
1, 1, 2, 3, 4, 5, 6, 7, 8
*SOLID SECTION, ELSET=ELALL, MATERIAL=SMA
*NSET, NSET=NOUT
3, 4, 7
*ELSET, ELSET=ELOUT
1
*NSET, NSET=TOP
3, 4, 7
*NSET, NSET=LEFT
1, 4, 8
*NSET, NSET=BOTTOM
1, 2, 5
*MATERIAL, NAME=SMA
*DENSITY
6450.0
*SPECIFIC HEAT
329.0
*CONDUCTIVITY
22.0
*DEPVAR
100
*USER MATERIAL, CONSTANTS=24
1.0, 3.0, 1.0E-8, 0.0, 16, 70.0E9, 30.0E9, 0.33,
22.0E-6, 22.0E-6, 291.0, 271.0, 295.0, 315.0, 0.05, -0.35E6,
-0.35E6, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1.0
*BOUNDARY
BOTTOM, 2, 2
BOTTOM, 5, 5
LEFT, 1, 1
*INITIAL CONDITIONS, TYPE=TEMPERATURE

```

```

NALL, 350.0
*STEP, INC=500
LOADING : TORSION
*STATIC
0.01, 1.0,, 0.01
*BOUNDARY
TOP, 5, 5, -1.5E-2
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=99999, POSITION=CENTROIDAL, SUMMARY=NO, ELSET=ELOUT
S11, S22, S33, S12, MISES, SDV2, SDV14, SDV15
*NODE PRINT, FREQUENCY=99999, SUMMARY=NO, NSET=NOUT
CF, U, RF
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL
TEMP, S, E
*END STEP
*STEP, INC=500
LOADING : COMPRESSION
*STATIC
0.01, 1.0,, 0.01
*BOUNDARY
TOP, 2, 2, 2.68E-5
*RESTART, WRITE, FREQUENCY=1
*EL PRINT, FREQUENCY=99999, POSITION=CENTROIDAL, SUMMARY=NO, ELSET=ELOUT
S11, S22, S33, S12, MISES, SDV2, SDV14, SDV15
*NODE PRINT, FREQUENCY=99999, SUMMARY=NO, NSET=NOUT
CF, U, RF
*OUTPUT, FIELD, FREQUENCY=1
*ELEMENT OUTPUT, ELSET=ELOUT, POSITION=CENTROIDAL
TEMP, S, E
*END STEP

```